History of Fractional Calculus to Fractional Derivatives

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Fractional calculus is a branch of mathematical analysis that extends the concept of differentiation and integration to non-integer (fractional) orders. This field has gained significant attention due to its ability to describe various complex physical and engineering phenomena more accurately than traditional integer-order calculus.

In 1695, the origin of fractional calculus dates back to the 17th century; the concept was introduced in a correspondence between Gottfried Wilhelm Leibniz and Guillaume de L'Hopital. In a letter to Leibniz on 30 September 1695, L'Hospital asked "what should be the result if $n=1/2$ in $d^n y/dx^n$?" Leibniz responded that "This is an apparent paradox, from which one day useful consequences will be drawn" (Miller, and Ross, 1993). Since then, fractional calculus, which originated from these early speculations, has gradually gained significant attention from researchers due to its extensive applications in various real-world fields. Many mathematicians contributed to the development of this branch of mathematical analysis, with pioneering work by Euler, Laplace, Abel, Liouville, and Riemann. Fractional calculus was mainly regarded as a purely theoretical topic for centuries, with little connection to the practical problems of physics and engineering. However, in the past three decades, it has emerged as an effective modeling methodology for researchers.

The concept and use of fractional derivatives provide evidence of a lengthy history of efforts by mathematicians and physicists to characterize the fine-scale behavior of natural events. Fractional derivatives generalize the concept of integer-order differentiation to non-integer orders. Common definitions include the Riemann-Liouville, Caputo, and Grünwald-Letnikov derivatives. Later, the idea of differential equations was presented in order to provide an explanation for actual nature problems. As a result, nonlinear differential equations of non-integer and fractional orders clearly reflect complicated physical phenomena found in nature.

Recent advancements in mathematical techniques have introduced fractional calculus as a powerful tool to better understand and describe these complex systems. Fractional derivatives, which generalize the concept of traditional integer-order derivatives to non-integer orders, offer a more nuanced approach to modeling the memory and hereditary properties of various materials and processes. In the context of optical fibers, incorporating fractional derivatives into the models of IASWs allows for a more accurate depiction of the wave dynamics and their interactions under different conditions. Fractional nonlinear evolution equations (FNLEEs) and their closed-form solutions are normally used to describe, model, and predict the underlying mechanisms in numerous fields, such as signal processing, control theory, system identification, solid-state physics, condensed matter physics, plasma physics, optical fibers, meteorology, oceanic phenomena, chemical kinetics, electrical circuits, quantum mechanics, biogenetics, fluid flow, and so on (Ahmad, 2020; Costa, 2019; Diethelm, 2010; Hilfer, 2000; Heisenberg, 2018; Kilbas et al., 2006; Lee, 2018; Mainardi, 1997; Miller et al., 1993; Podlubny, 1999; Silva, 2022; Vitanov, 2022; Vong et al., 2010; Zhou, 2009). Most evolution equations carry the nonlinear characteristics of physical phenomena. The ambiguity of numerous physical problems closely related to real-life situations is illuminated by pictures of solutions to fractional-order nonlinear evolution equations (FNLEEs). Therefore, looking for exact solutions to fractional-order nonlinear evolution equations has been popular in the real life field.

A key component of fractional calculus is the fractional derivative. Fractional derivatives extend the concept of differentiation to non-integer orders, allowing us to describe and analyze processes that exhibit complex behaviors such as anomalous diffusion, viscoelasticity, and fractal phenomena. Unlike integer-order derivatives which represent rates of change over whole numbers of dimensions, fractional derivatives can describe rates of change over fractional dimensions. This makes them essential in fields like physics, engineering, and applied mathematics, where systems exhibit non-local or memory-dependent behaviours. They are defined through various mathematical formulations, such as the Riemann-Liouville, Caputo, and Grünwald-Letnikov definitions, each suited to different types of problems and boundary conditions. Examples of fractional derivatives include the Riemann-Liouville fractional derivative, the Caputo fractional derivative, the Caputo-Fabrizio derivative, the Atangana derivative, the Hadamard derivative, and the Beta derivative. Fractional derivatives have found applications in diverse areas including signal processing, control systems, biomedical engineering, and finance which highlights their versatility and importance in modeling real-world phenomena accurately.

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